A comprehensive exam report on ”Decoherence” by Maximilian A. Schlosshauer.

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1 Intro

In undergraduate and often times in graduate courses on quantum mechanics, quantum mechanical systems are considered to be closed. This assumption is made tacitly by assuming that Schrödinger’s equation governs all quantum phenomena. Schrödinger’s equation is an equation that generates unitary evolution, this will be seen later, as a result there is no dissipation or decoherence. But decoherence and dissipation are inevitable, every quantum system will experience both of these environmental disturbances. There do exist interactions which may be approximately described by a closed system with high precision, but in these cases decoherence, a fundamental quantum mechanical process that occurs when a quantum system is measured/interacted with, is explained only by the Copenhagen interpretation of quantum mechanics which says that upon being measured a quantum system instantaneously collapses to a definite state within the Hilbert space associated with the quantum system. This occurs with some probability distribution governing the "collapse". This is at odds with the seemingly continuous world we live in and one might guess that the Copenhagen school of thought has missed something fundamental. Indeed decoherence is not a phenomenon that occurs instantaneously, decoherence time scales which vary from system to system can actually be computed if one considers the more realistic setting where the system in question, referred to as $S$ in this paper, and the associated environment, referred to as $E$, are both treated as a quantum system and the system of interest $S$ sits within $E$, interacting with $E$ via some Hamiltonain. By considering a larger system one may use our familiar Schrödinger equation as a starting point and later deduced the reduced dynamics of $S$. Ideally we hope to obtain the unitary evolution for the total dynamics and deduced the dynamics local to $S$ but, as will be seen in what is to come, this is no easy venture. We will develop methods for deducing the reduced dynamics and motivate with various examples. We shall also introduce methods to safeguard against decoherence. Finally, some experiments that prove the continuous exponential decay of quantum coherences and their susceptibility to environmental monitoring will be discussed.

2 Decoherence in a nutshell

Schrödinger’s equation for an isolated quantum system is

$$i\hbar \partial_t |\psi(t)\rangle_S = H_S |\psi(t)\rangle_S$$

where we subscript the state vector of the system and the self Hamiltonian with $S$. $|\psi(t)\rangle \in \mathcal{H}_S$ a separable Hilbert space, can be finite or infinite. Assuming that $H_S$ is time independent, the solution of the differential equation above is simply

$$|\psi(t)\rangle_S = U(t)|\psi(0)\rangle_S$$

where

$$U(t) = e^{-i\frac{\mathcal{H}_S t}{\hbar}}.$$  

This operator is referred to as the time evolution operator and $|\psi(0)\rangle_S$ is the initial state of the system. The superposition principle says that the general form of our initial state can take is the following.

$$|\psi(0)\rangle_S = \sum_i c_i |\phi_i\rangle_S$$

where $c_i$ are complex numbers and $\{|\phi_i\rangle\}_i$ is an orthonormal basis for the system’s Hilbert space, call it $\mathcal{H}_S$, which are often the eigenvectors of some Hermitian operator over the $\mathcal{H}_S$. It is important to note that even if initially our state is of the form $|\psi(0)\rangle_S = c_k |\phi_k\rangle$, upon evolving the state with $U(t)$, the evolved state will in general be a superposition.

The system’s state yields the following probability density function

$$|\langle \phi_i |\psi(t)\rangle|^2$$

which tells us the probability of measuring the value $\phi_i$ when conducting a measurement on the corresponding observable, some Hermitian operator $X_\phi$. Unfortunately this formalism does not give us the means to monitor the evolution of the non-classicality of quantum states. We have no explicit way to monitor the dynamics of the superposition quality of our system’s state. If monitoring the latter is desired we may choose to analyze the density matrix of our system which yields a more explicit view of how the non-classicality of a quantum system evolves with time. If the system is in a pure state at time $t$, i.e. $|\psi(t)\rangle = c(t)|\phi_i\rangle$ then the density matrix of the system may be constructed by simply taking the outer product of the state vector with itself.

$$|\psi(t)\rangle\langle \psi(t)| = |c(t)|^2 |\phi_i\rangle\langle \phi_i|$$

this state in the matrix representation using the basis $\{|\phi_i\rangle\}$ is simply a matrix $\rho_{ij}$ of all zeros with the exception of the $A_{ii}$
component which is of course equal to $|c(t)|^2$.

$$
|\psi(t)\rangle\langle\psi(t)| \rightarrow \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & |c_i(t)|^2 & \vdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots 
\end{pmatrix}
$$

(7)

Now, if we have a more complex situation like the earlier instance where $|\psi(t)|_S$ has the superposition quality then the density matrix in this case has the following components $\rho_{ij} = \langle \phi_i | \psi(t) \rangle \langle \psi(t) | \phi_j \rangle = c_i(t)c_j^*(t)$. Or in matrix form

$$
|\psi(t)\rangle\langle\psi(t)| \rightarrow \begin{pmatrix}
|c_1(t)|^2 & c_1^*c_2^*(t) & \cdots & c_1^*c_i^*(t) & c_1^*c_{i+1}^*(t) & \cdots \\
c_2c_1^*(t) & |c_2(t)|^2 & \cdots & c_2^*c_i^*(t) & c_2^*c_{i+1}^*(t) & \cdots \\
c_3c_1^*(t) & c_3c_2^*(t) & \cdots & |c_3(t)|^2 & c_3^*c_{i+1}^*(t) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
c_i^*c_1^*(t) & c_ic_2^*(t) & \cdots & |c_i(t)|^2 & \vdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots 
\end{pmatrix}
$$

(8)

The off-diagonal terms of this matrix are in direct correlation with the superposition quality. The off-diagonal terms of the density matrix are called coherence terms and they quantify the non-classicality of a quantum system. If indeed these coherence terms vanish by some means then the density matrix of our system yields a classical probability structure characterized by the probability density in equation (4). In (2.1) we will elaborate on these quantum coherences for the case of closed systems and will later introduce quantum open systems in (2.2).

### 2.1 Example of quantum coherence for a closed system.

The simplest class of quantum systems are two-level systems. A two-level system can be successfully used to model spin $\frac{1}{2}$ particles, spontaneous decay of excited atoms and even tunneling in the ammonia molecule just to give some examples. In what follows we will discuss the mathematical structure pertaining to two-level systems via a study of the ammonia molecule.

#### 2.1.1 The Ammonia molecule.

To keep things simple we will assume that the ammonia molecule we will study to be a closed system. Although closed systems do not actually exist in nature many systems can be approximated by closed systems with a high degree of fidelity. Well therefore ignore any interactions between the molecule and its environment. The ammonia molecule has an equilibrium structure which is depicted in figure 1. Ignoring the internal dynamics as well, rotational dynamics and translational dynamics, the degrees of freedom of this molecule are reduced to the orientation of the nitrogen atom with respect to the plane formed by the hydrogen atoms. The hydrogen atom can be either above this plane $|1\rangle$ or below it $|2\rangle$. Due to the pyramidal symmetry of the ammonia molecule we can conclude that both of these states have the same energy. The diagonal components of this system’s Hamiltonian are therefore $\langle 1 | H | 1 \rangle = \langle 2 | H | 2 \rangle = E_0$. Due to quantum tunneling the nitrogen atom has a non zero transition probability to go from the $|1\rangle$ state to the $|2\rangle$ and vice versa. Let us give this transition probability the value $-\epsilon$. Thus, $\langle 1 | H | 2 \rangle = -\epsilon$ and due to Hermiticity of the Hamiltonian $\langle 2 | H | 1 \rangle = -\epsilon$. The matrix representation of this Hamiltonian is as follows.

$$
H \rightarrow \begin{bmatrix}
E_0 & -\epsilon \\
-\epsilon & E_0
\end{bmatrix}
$$

(9)
Diagonalizing this matrix is very simple. The eigen vectors and corresponding eigen values are

\[ |E_\pm \rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle), \quad E_\pm = E_0 \pm \epsilon. \]  

(10)

Let us assume that the initial state of this ammonia molecule is \( |\psi(0)\rangle = |1\rangle \). The state of the system at some time \( t \) will then be \( U(t)|\psi(0)\rangle = e^{\frac{iE\epsilon}{\hbar}t}|\psi(0)\rangle \). Let us expand this out and take a closer look at what this state looks like. We first rewrite \( |1\rangle \) as \( \frac{1}{\sqrt{2}}(|E_-\rangle + |E_+\rangle) \) and then apply the time evolution operator.

\[ |\psi(t)\rangle = e^{-\frac{iE\epsilon}{\hbar}t} \left( |E_-\rangle + |E_+\rangle \right) = e^{-\frac{iE\epsilon}{\hbar}t} \left( \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) + e^{\frac{2iE\epsilon}{\hbar}t} \left( |1\rangle - |2\rangle \right) \right) = e^{-\frac{iE\epsilon}{\hbar}t} \left( \frac{1}{\sqrt{2}} |1\rangle + (1 + e^{\frac{2iE\epsilon}{\hbar}t}) |2\rangle \right) = e^{-\frac{iE\epsilon}{\hbar}t} \left( \cos\left(\frac{2Et}{\hbar}\right) |1\rangle + i\sin\left(\frac{2Et}{\hbar}\right) |2\rangle \right). \]

(11)

The density matrix is therefore

\[ \rho(t) = \frac{1}{2} \left[ \begin{array}{cc} \cos^2\left(\frac{\epsilon t}{\hbar}\right) & -\cos\left(\frac{\epsilon t}{\hbar}\right) \sin\left(\frac{\epsilon t}{\hbar}\right) \\ i\sin\left(\frac{\epsilon t}{\hbar}\right) \cos(\epsilon t) & \sin^2\left(\frac{\epsilon t}{\hbar}\right) \end{array} \right]. \]

(12)

The important thing to note here is that the off-diagonal terms are periodic. Although there are \( t \) values that make this matrix diagonal the coherences are almost always present. This is a characteristic unique to closed systems. Loss of coherence, decoherence, only takes place when a measurement is conducted to the system and the wave function collapses.

The measurement postulate as described by Nielsen and Chuang [1] tells us that measurements are described by a collection of projector operators \( M_m \) where the index \( m \) represents the observable values attainable from such measurements. If we conduct a projective measurement on a state \( |\psi(t)\rangle \) and measure \( m \) then the resulting state is

\[ \frac{M_m|\psi(t)\rangle}{\sqrt{\langle\psi(t)|M_m^\dagger M_m|\psi(t)\rangle}}. \]

(13)

The probability of observing \( m \) is indeed \( p(m) = \langle \psi(t)|M_m^\dagger M_m|\psi(t)\rangle \). The non unitary evolution summarized as follows.

\[ |\psi\rangle \xrightarrow{\text{measurement}} \frac{M_m|\psi(t)\rangle}{\sqrt{\langle\psi(t)|M_m^\dagger M_m|\psi(t)\rangle}}. \]

(14)

Assuming we have a measurement device that can measure the location of the nitrogen atom in our ammonia molecule our set of projective measurement operators is then \( \{|1\rangle, |2\rangle\} \). The probabilities of finding the atom in the state \( |1\rangle \) and \( |2\rangle \) are \( p(1) = \cos^2\left(\frac{\epsilon t}{\hbar}\right) \) and \( p(2) = \sin^2\left(\frac{\epsilon t}{\hbar}\right) \) respectively. If we make a measurement and observe \( |1\rangle \) then the state \( |\psi(t)\rangle \) of course collapses to the state \( |1\rangle \) with the use of (14), in density matrix notation

\[ \left[ \begin{array}{cc} \cos^2\left(\frac{\epsilon t}{\hbar}\right) & -\cos\left(\frac{\epsilon t}{\hbar}\right) \sin\left(\frac{\epsilon t}{\hbar}\right) \\ i\sin\left(\frac{\epsilon t}{\hbar}\right) \cos(\epsilon t) & \sin^2\left(\frac{\epsilon t}{\hbar}\right) \end{array} \right] \xrightarrow{\text{measurement}} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]. \]

(15)

similarly, measuring the ammonia molecule to be in the \( |2\rangle \) state is summarized as follows.

\[ \left[ \begin{array}{cc} \cos^2\left(\frac{\epsilon t}{\hbar}\right) & -\cos\left(\frac{\epsilon t}{\hbar}\right) \sin\left(\frac{\epsilon t}{\hbar}\right) \\ i\sin\left(\frac{\epsilon t}{\hbar}\right) \cos(\epsilon t) & \sin^2\left(\frac{\epsilon t}{\hbar}\right) \end{array} \right] \xrightarrow{\text{measurement}} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right]. \]

(16)

The most important thing to take note of here is that the off-diagonal entries are completely and instantaneously wiped out in both cases. This non-unitary jump destroys the non-classicality of our quantum state. We have complete and instantaneous decoherence due to measurement, these type of measurements are sometimes referred to as strong — measurements and are the center piece of the celebrated Copenhagen interpretation of quantum mechanics. In what follows, we will see that moving to an open system setting also renders decoherence but in a continuous manner as opposed the instantaneous decoherence we have just seen in the ammonia molecule system. As mentioned earlier, closed systems do not actually exists, approximating our quantum system with a closed system corners us into a situation like our ammonia molecule model where we had to use strong-measurements in order to interpret measurement. However, if we consider our system to be a dynamic body within some environment that is assumed inherently quantum a better understanding of how decoherence takes place continuously within a finite time frame can be obtained.
2.2 Partial trace

To treat compound quantum systems one need only construct bigger Hilbert spaces. If we have a quantum system whose state, $|\psi_S\rangle$ evolves within the space $\mathcal{H}_S$ and interacts with another quantum system whose state, $|\psi_E\rangle$ evolves within the Hilbert space $\mathcal{H}_E$ then the total state of the compound system to be in the space $\mathcal{H}_S \otimes \mathcal{H}_E$. The most general state that describes the state of the compound quantum system in question will be of the form

$$\sum_{i=1}^{\min\{n,m\}} \alpha_i |\psi_i\rangle \otimes |\phi_i\rangle$$  \hspace{1cm} (17)

where $\{|\psi_i\rangle\}_{i=1}^m$ and $\{|\phi_i\rangle\}_{i=1}^n$ are orthonormal basis (ONB) for $H_S$ and $H_E$ respectively. This is called a Schmidt decomposition [1].

2.2.1 The environment as a quantum system

Let us take the environment to be the system, also quantum, whose state evolves in the Hilbert space $\mathcal{H}_E$, $\text{dim}(\mathcal{H}_E) = n$ with ONB $\{|E_i\rangle\}_i$. The "system" $S$ will be a quantum system whose state lives in the Hilbert space $\mathcal{H}_S$, $\text{dim}(\mathcal{H}_S) = m$ with ONB $\{|\psi_i\rangle\}_i$. The total state of the compound system will therefore live in $\mathcal{H}_S \otimes \mathcal{H}_E$. Compound quantum systems of this form are called "quantum open systems" if the dynamics is generated by a Hamiltonian $H_{SE} = H_S + H_E + H_I$ which includes the interaction dynamics between the system and the environment $H_I$. Ignoring the dynamics let $|\psi_{SE}(0)\rangle$ be some initial state, then in general it will have the form

$$|\psi_{SE}(0)\rangle = \sum_{i=1}^m \alpha_i |\psi_i\rangle \otimes |E_i\rangle.$$  \hspace{1cm} (18)

The corresponding density matrix is

$$|\psi_{SE}(0)\rangle \langle \psi_{SE}(0)| = \rho_{SE} = \sum_{i} \sum_{j} \alpha_i \alpha_j^* \langle \psi_i | \psi_j \rangle \otimes |E_i \rangle \langle E_j |.$$  \hspace{1cm} (19)

Here we have the state of the total system. However, if we are only interested in how $S$ evolves with time, or what the local state of the system $S$ is, we need a way to deduced the local dynamics. How do we obtain the state matrix pertaining to just the system $S$? It turns out that we need to trace out the degrees of freedom pertaining to the environment ([2] chapter 2). The resulting matrix, $Tr_E{\rho_{SE}(0)}$, is the state matrix of the system $S$, call it $\rho_S$. The partial trace $Tr_E$ is defined as follows.

**Definition 1.** $Tr_E:\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_S)$ The partial trace $Tr_E$ is a mapping from the density matrices $\rho_{SE} = |\psi_{SE}\rangle \langle \psi_{SE}|$ in the Banach space $\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E)$ of trace-class operators on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$ to $\rho_S \in \mathcal{B}(\mathcal{H}_S)$.

$$Tr_E{|\psi_{SE}\rangle \langle \psi_{SE}|} := \sum_i \langle \phi_i | \psi_{SE} \rangle \langle \psi_{SE} | \phi_i \rangle$$

$\{|\phi_i\rangle\}$ is some ONM of $\mathcal{H}_E$, it can be shown that the partial trace is basis independent.

This mapping yields the appropriate reduced dynamics that allows us to see how the environment $E$ affects our system $S$ without having to keep track of what the environment is doing. Here is the partial trace of (19)

$$Tr_E{\rho_{SE}(0)} = \sum_k \sum_i \sum_j \alpha_i \alpha_j^* \langle \psi_i | \psi_j \rangle \langle E_k | E_i \rangle \langle E_j | E_k \rangle =$$

$$= \sum_k \sum_i \sum_j \alpha_j \alpha_i^* \langle \psi_k | \psi_i \rangle \langle E_k | E_i \rangle =$$

$$= \sum_k |\alpha_i|^2 |\psi_k\rangle \langle \psi_k|.$$  \hspace{1cm} (20)

This is an astonishing results for two reasons. First, the resulting matrix is diagonal. Taking the partial trace over the perfectly distinguishable environmental degrees of freedom has induced optimal decoherence. Second, the resulting density matrix is a "mixed" state. This is critical because the total state matrix prior to the partial tracing was a "pure" state. With this "reduced" density matrix we can compute expectation values of any local observable, $A_S$ (Hermitian operator) over $\mathcal{H}_S$, as follows

$$\langle A_S \rangle = Tr_S{\rho_S A_S}.$$  \hspace{1cm} (21)
To reassure ourselves that the partial trace \( \text{Tr}_E \) is the appropriate mapping to use in order to deduce the local dynamics lets consider an arbitrary observable \( A_S \) over \( \mathcal{H}_S \), there is a natural embedding of such an observable that extends it to the space of observables over \( \mathcal{H}_S \otimes \mathcal{H}_E \).

\[
A_S \rightarrow A_S \otimes I_E,
\]

where \( I_E \) is the identity matrix in the \( \mathcal{H}_E \). In quantum open systems one primarily interest themselves in the statistical properties pertaining to observables of the form \( A_S \otimes I_E \). One can obtain the expectation value of such an observable by using the total system’s state matrix as follows

\[
\langle A_S \otimes I_E \rangle = \text{Tr}\{\rho_{SE}(A_S \otimes I_E)\}.
\]

But it can be shown that

\[
\text{Tr}\{\rho_{SE}(A_S \otimes I_E)\} = \text{Tr}_S\{\rho_S A_S\}.
\]

This means that precisely all of the information need to compute the statistical properties of some observable in the system \( S \) is contained in the reduced density matrix \( \rho_S \).

### 2.2.2 Environment induced non-unitarity

Let \( H_{SE} \) be the Hamiltonian governing the dynamics of some quantum open system. Assuming that the Hamiltonian is time independent, we can immediately make form of the time evolution operator of the compound system. i.e. \( U = e^{-\frac{i}{\hbar} H_{SE}} \).

Assuming that the system and the environment are uncorrelated at \( t = 0 \), i.e. \( \rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0) \) and evolving the state matrix using the time evolution operator as well as tracing out the environmental degrees of freedom we arrive at the following

\[
\rho_S(t) = \text{Tr}_E[U(t)(\rho_S(0) \otimes \rho_E)]U^\dagger(t)].
\]

Now, assuming that the environment is in the state \( \rho_E(0) = \sum_i p_i |E_i\rangle\langle E_i| \) at \( t = 0 \) the above reduces to

\[
\rho_S(t) = \sum_{ij} p_i \langle E_j|U(t)|E_i\rangle \rho_S(0)\langle E_i|U^\dagger(t)|E_j\rangle.
\]

The operators \( \langle E_j|U(t)|E_i\rangle \) are called Kraus operators and these operators evolve the state of the system \( S \) non-unitarily. The Kraus operators carry information about the environments initial state and the dynamics of the joint system \( SE \), furthermore the map \( \Phi := \sum_{ij} p_i \langle E_j|U(t)|E_i\rangle |...|E_i|U^\dagger(t)|E_j\rangle \) is a trace preserving completely positive map and these Kraus operators have the property \( \sum_{ij} \langle E_j|U(t)|E_i\rangle \langle E_i|U^\dagger(t)|E_j\rangle = I_S \). The non-unitarity can present itself explicitly in two ways.

- The first is decoherence. The decaying of the off-diagonal elements in the reduced state matrix \( \rho_S(t) \). Quantum coherence is very delicate and if there is an environment interacting with our system then the system is essentially being "measured" continuously by said environment and measurements eliminate quantum coherence. We deliberately use the term measurement in order to connect this description of decoherence to that of the Copenhagen interpretation.

- The second is dissipation, it is a common mistake to consider dissipation and decoherence as analogous or even synonymous but this is incorrect. Indeed one can have decoherence without dissipation even though dissipation is always accompanied by decoherence. A model that exhibits decoherence but no dissipation is recoilless scattering, this will be treated in chapter 4. This is a very interesting model because although the particles self dynamics are taken to be that of a free particle unhindered by the environmental scattering, the system nevertheless undergoes exponential decoherence in in the position basis due to the interaction with the environment.

### 2.3 Environment-induced superselection

So far we have interested ourselves in the dynamics of both open and closed quantum systems. In each instance we have have selected a basis for our system and/or environment respectively without much scrutiny or without elucidation of why the respective basis were chosen. The nature of the initial state of a quantum system is going to play a critical role in the dynamics of the coherences. A very important case occurs when the initial state of the system is in a superposition of eigen states of an observable \( O_S \) that commutes with the system-environment Hamiltonian \( H_{SE} \), i.e.

\[
0 = [O_S, H_{SE}] = \sum_i a_i |o_i\rangle\langle o_i|, H_{SE} = \sum_i a_i |o_i\rangle\langle o_i|, H_{SE} = 0,
\]

meaning that the states \( |o_i\rangle \) will also be eigen states of \( H_{SE} \), if we embed them into the larger Hilbert Space \( \mathcal{H}_A \otimes \mathcal{H}_B \). These basis are referred to as pointer basis and the operator they form an eigen basis for is referred to as a pointer observable. As an example let us consider the case where \( H_{SE} = H_S + H_E + H_I \approx H_I \). This case is referred to as the quantum –
It is clear that the system completely uncorrelated at

It must be mentioned that this is a very specific class of initial conditions, they assume that the environment and system are

have a separable initial total state \( \rho_{SE}(0) = |\alpha_i\rangle \otimes \rho_E(0) \) then the system \( S \) will be completely immune to the dynamics.

\[
Tr_E \{ |\alpha_i(t)\rangle \langle \alpha_i(t) | \} = Tr_E \{ e^{\frac{i}{\hbar} H_I t} \rho_{SE}(0) \otimes \rho_E(0) e^{-\frac{i}{\hbar} H_I t} \} = \rho_S(0) Tr_E \{ e^{\frac{i}{\hbar} H_I t} \rho_E(0) e^{-\frac{i}{\hbar} H_I t} \} = \rho_S(0).
\]

Now let us take it a step further and see how \( H_I \) affects an open system with the system in a superposition state with respect to the basis \( \{ |\alpha_i\rangle \} \) at \( t = 0 \),

\[
\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0) = \sum_{ij} c_{ij} |\alpha_i\rangle \langle \alpha_j | \otimes \rho_E(0).
\]

It must be mentioned that this is a very specific class of initial conditions, they assume that the environment and system are completely uncorrelated at \( t = 0 \). Now, let us evolve this state.

\[
\rho_{SE}(t) = e^{-\frac{i}{\hbar} H_I t} \rho_{SE}(0) e^{\frac{i}{\hbar} H_I t} = e^{-\frac{i}{\hbar} O_{\sum} E_i} \sum_{ij} c_{ij} |\alpha_i\rangle \langle \alpha_j | \otimes \rho_E(0) e^{\frac{i}{\hbar} O_{\sum} E_i} =
\]

\[
= \sum_{ij} c_{ij} |\alpha_i\rangle \langle \alpha_j | \otimes e^{-\frac{\hbar}{i} E_i t} \rho_E(0) \langle \alpha_n | \otimes \sum_m e^{\frac{\hbar}{i} E_m t} E_m =
\]

\[
= \sum_{jk} c_{jk} |\alpha_j\rangle \langle \alpha_k | \otimes e^{-\frac{\hbar}{i} E_j t} \rho_E(0) \otimes \sum_m e^{\frac{\hbar}{i} E_m t} E_m.
\]

It is clear that the system \( S \) has remained unchanged! However, notice that the system is now entangled with its environment, i.e. it is no longer a separable state like it was at \( t = 0 \). To explicitly see the contribution the environment adds to the dynamics of the system \( S \), we now perform a partial trace over the environment.

\[
\rho_S(t) = Tr_E[\rho_{SE}(t)] = \sum_{jk} c_{jk} \sum_n \langle E_n | \otimes e^{-\frac{\hbar}{i} E_j t} \rho_E(0) \otimes e^{\frac{\hbar}{i} E_m t} \rho_E(0) = 
\]

\[
= \sum_{ij} c_{ij} \sum_n \langle E_n | \otimes e^{-\frac{\hbar}{i} E_i t} \rho_E(0) \otimes e^{\frac{\hbar}{i} E_m t} \rho_E(0) = 
\]

Notice that the diagonal elements of the above matrix have remained unchanged! However, the off-diagonal elements are evolving in time and are \( c_{ij} \langle \psi_{E_i}(t) | \psi_{E_j}(t) \rangle \). Since our system was in a superposition at \( t = 0 \) it is of no surprise that there will be quantum coherence present but what is intriguing is the fact that the coherences are evolving in a manner prescribed by \( H_I \). The only property of the system \( S \) being affected are the quantum coherences. This indeed illustrates a general case where decoherence takes place without dissipation. For quantum open systems we generally expect these coherences to decay exponentially, convincing evidence of this will be provided in the following chapters; even if we are really careful in the selection of our initial state, a superposition of eigen states of the interaction Hamiltonian for example, we still have decoherence. If we wanted to be more careful and avoid/reduce decoherence as well this is too is possible by using decoherence free subspaces which suffer from more stringent requirements than pointer basis do. We will come bad to decoherence free subspaces in a later section.

### 2.3.1 Finding the appropriate pointer basis

Above we have explored a special where the dynamics of an open quantum system are governed largely by the interaction Hamiltonian \( H_I \), i.e. \( H_{tot} \approx H_I, \text{strong coupling limit} \), using the quantum measurement limit, i.e. \( H_I = \rho_S \otimes \sum_i E_i \). In such a case we need only find states that are simultaneous eigen states of both the interaction Hamiltonian and the appropriate pointer observable. In this limit the pointer observable is indeed \( \rho_S \). This is all in an approximate senses as the intrinsic dynamics of the system and the environment are being ignored. The approximation will indeed loose fidelity if the system is allowed to evolve for large enough times that enter the regime where the self Hamiltonian of the system \( S \) and or environment \( E \) become consequential. If the interaction Hamiltonian does not align to the quantum measurement limit then more work needs to be done in order to deduce the pointer basis.
The opposite limit to that presented above is the case where the interaction Hamiltonian is taken to be relatively weak and slowly varying with respect to the system’s self Hamiltonian, we will refer to this limit as the weak – coupling limit. In this case it is clearly the case that the pointer states will virtually be the eigen state of the system’s Hamiltonian, i.e. \( H_{\text{tot}} \approx H_S \). This are simply the energy eigen states of the respective system and will be so regardless of the form of interaction it undergoes with its environment. Again, this in in an approximate sense and if the time is evolved to a regime where the interaction Hamiltonian is effective then the approximation will no longer hold water.

Unfortunately the strong – coupling and the weak – coupling limits are not always applicable. What can be done for everything in between? To answer this question first we need to introduce purity measures \([1]\).

**Definition 2. Purity measure**

\[
\xi(\rho_S) := \text{Tr}(\rho_S^2)
\]

\[
\frac{1}{N} \leq \xi(\rho_S) \leq 1 \quad \text{where } N := \text{dim}(H_S)
\]

A measure of this sort is used to measure the purity of a quantum state. The case where \( \xi(\rho) = \frac{1}{N} \) pertains to the maximally mixed case. Now, if \( \xi(\rho) = 1 \) then \( \rho \) is a pure state and \( \rho^2 = \rho \). This measure can also be used to track the purity of a quantum state undergoing non-unitary evolution. In the case of quantum open systems, \( \xi(\rho(t)) \) is expected to gravitate away from 1 as soon as the clock starts ticking. To see this consider the reduced state of a general open quantum system as derived in (34) using the strong – coupling and quantum-measurement limits.

\[
\rho_S(t) = \sum_{ik} c_{ik} \langle \psi_{E_k}(t) | \psi_{E_k}(t) \rangle |a_i\rangle \langle o_k|.
\]

We now compute the purity measure define above defining \( a_{ik}(t) := c_{ik} \langle \psi_{E_k}(t) | \psi_{E_k}(t) \rangle \).

\[
\xi(\rho_S(t)) = \text{Tr}\{ \sum_{lk} a_{lk} a_{km} |o_l\rangle \langle o_m| \}.
\]

Taking the trace we get the following,

\[
\xi(\rho_S(t)) = \sum_{lk} a_{lk}(t) a_{kl}(t).
\]

At \( t = 0 \) this is just \( \sum_{ik} c_{ik} c_{kl} = 1 \). For large enough \( t \) \( \xi(\rho_S(t)) = \sum_{k} c_{kk}^2 \). All of the \( c_{kk} \) are positive and less than one, also recall that \( \sum_{k} c_{kk} = 1 \). Therefore, \( \sum_{k} c_{kk}^2 < 1 \). This fact is also true for small values of \( t \), so that the moment the clock starts to tick and the dynamics take place our system’s state loses purity.

Another important purity measure is the Von Neuman Entropy.

**Definition 3. Von Neuman Entropy**

\[
S_V(\rho_S) = -\text{Tr}(\rho_S \ln \rho_S)
\]

\[
0 \leq S_V(\rho_S) \leq \ln(N) \quad \text{where } N \text{ is the dimension of the Hilbert space } H_S
\]

This measure can also serve as a measurement of purity, \( S = 0 \) pertains to a pure state and \( S = \ln(N) \) pertains to a maximally mixed state. The final measure that we present is an approximate version of the Von Neuman Entropy called the Linear Entropy.

**Definition 4. Linear Entropy**

\[
S_L(\rho_S) = \frac{N}{\ln(N)} (1 - \text{Tr}(\rho_S^2))
\]

\[
0 \leq S_L(\rho_S) \leq 1 \quad \text{where } N \text{ is the dimension of the Hilbert space } H_S.
\]

Here \( S_L \) for pure states and \( S_L = 1 \) for maximally mixed states. All of this measures are related to one another by the fact that \( S_L \) is a first order approximation to \( S \).

We now return to our discussion on pointer states and their deduction in general. Using the Von Neuman entropy one can gauge the degree of immunity an initial system state has to the total dynamics induced by the corresponding Hamiltonian \( H_{SE} \). Focusing only on separable states for a moment let our initial state be something of the form \( \rho(0) = \rho_S(0) \otimes \rho_E(0) \). Then the Von Neuman entropy of the reduced state of the system \( \rho_S(t) = \text{Tr}_E(U(t) \rho_S(0) \otimes \rho_E(0) U(t)^\dagger) \) is the following

\[
S_V(\rho_S(t)) = -\text{Tr}_S[\rho_S(t) \ln \rho_S(t)].
\]

It is suggested in the literature, Zurek’s 1994 paper in particular \([3]\), that one test all possible pure states, such a method is referred to as the ”predictability sieve”. The pure states that yield the smallest entropy after being evolved in time will form the pointer basis. This seems to lack rigor since there is no dimension to the pointer basis specified, or a cutoff value for the entropy, but as Zurek puts it \([3]\) "Where exactly the quantum-classical border is erected is a subjective matter, to be decided by circumstances”. This sieve is indeed very computationally complex and unfortunately impossible to carry out in cases where the Hilbert spaces are infinite or extremely large. In cases where the \( S_L \) is approximately \( S_V \) the computational challenge is drastically reduced since one no longer has to diagonalize the density matrix in order to compute \( \ln \rho_S \). It is interesting to note the using the predictability sieve Zurek in his paper \([3]\) is able to conclude that the pointer states of a harmonic oscillator coupled to a heat bath are the coherent states. This hints at the fact that the pointer states will in general form and over complete set confined to a subset of the systems Hilbert space \( \mathcal{H}_S \).
2.4 decoherence-free subspaces

Pointer states are robust against decoherence but superposition of pointer states are not and will in general decohere very rapidly. An important problem in quantum information processing, quantum computing and quantum technologies is minimizing decoherence. The latter means that given a pointer basis we must find a sub space that is the most immune to decoherence, such spaces are referred to as decoherence-free subspaces (DFS). Let us consider a quantum open system in the quantum measurement limit. A more general interaction Hamiltonian than what has already been presented is $H_I = \sum_i S_i \otimes E_i$. The pointer basis in this case is simply the set of simultaneous eigen states of the operators $S_i$. The DFS corresponding to this system will be spanned by an appropriate sub basis of this pointer basis. The restriction required will be that all elements of the DFS be simultaneous degenerate eigen states of the operators $S_i$ corresponding to this system will be spanned by an appropriate sub basis of this pointer basis. The restriction required will be that all elements of the DFS be simultaneous degenerate eigen states of the operators $S_i$. Assuming $\{|s_i\}_i$ is a DFS let $|\psi_S(0)\rangle = \sum_i c_i |s_i\rangle$. Take the initial state of some open system to be $|\psi_{tot}(0)\rangle = \{\sum_i c_i |s_i\rangle\} \otimes |\psi_E(0)\rangle$ and let us now evolve this system to obtain

$$|\psi_{SE}(t)\rangle = \sum_i e^{-it \frac{H_I}{\hbar}} c_i |s_i\rangle \otimes |\psi_E(0)\rangle = \sum_i e^{-it \frac{\sum_j \lambda_j s_i \otimes E_j}{\hbar}} c_i |s_i\rangle \otimes |\psi_E(0)\rangle = \sum_i c_i |s_i\rangle \otimes e^{-it \frac{\sum_j \lambda_j s_i \otimes E_j}{\hbar}} |\psi_E(0)\rangle = |\psi_s(0)\rangle \otimes |\psi_E(t)\rangle. \quad (35)$$

If we now switch to the density matrix representation and trace out the environmental degrees of freedom it is immediately clear that no decoherence takes place.

$$\rho_S(t) = Tr_E(|\psi_{SE}(t)\rangle \langle \psi_{SE}(t)|) = Tr_E(|\psi_S(0)\rangle \langle \psi_S(0)| \otimes |\psi_E(t)\rangle \langle \psi_E(t)|) = \rho_S(0) Tr_E(|\psi_E(t)\rangle \langle \psi_E(t)|) = \rho_S(0). \quad (37)$$

If DFS are present and they are large then one is capable of storing quantum information as elements in the span of these subspaces and rest at ease with the knowledge that the environment in which such states lie will not disturb (significantly in the realistic approximate case) the stored information. Unfortunately DFS are not at all ubiquitous and very restricting symmetries will have to be in place for them to be present. When possible one my decompose $H_{SE}$ or $H_I$ in the quantum measurement limit into a two parts, one containing some nice symmetries that yield DFS and a small perturbation that breaks the symmetry, i.e. $H_I \approx H_{DFS} + H_{no-DFS}$, $H_{DFS}$ being a Hamiltonian for which DFS can be found, and $H_{no-DFS}$ is some negligible perturbation. In the case where DFS can not be found, this is almost all cases unfortunately, approximate DFS form the states that are the most most robust to decoherence effects.

2.4.1 Symmetric environmental dephasing

To illustrate what finding a DFS looks like let us consider a very simple system of $N$ qubits coupled to some environment in the following symmetric way.

$$|0\rangle_j \rightarrow |0\rangle_j \quad (38)$$

$$|1\rangle_j \rightarrow e^{i\phi} |1\rangle_j. \quad (39)$$

This model follows closely the work of Daniel A. Lidar in his paper of DFS [4] Where $j$ is used to index over all of the qubits. For example, if $N = 2$ a state $|\psi\rangle = |01\rangle$ will evolve to $e^{i\phi} |01\rangle$ via interaction with the environment. Where $\phi$ varies depending on the environmental mode the spins are coupled to. Let us now assign a probability $p_\phi$ for each dephasing process. Let the initial state of a system of $N$ spins be

$$|\psi\rangle_0 = \bigotimes_{j=1}^N (a_j |0\rangle_j + b_j |1\rangle_j),$$

then the random process of interacting with the environmental mode corresponding to $\phi$ outputs the following state.

$$|\psi\rangle_\phi = \bigotimes_{j=1}^N (a_j |0\rangle_j + b_je^{i\phi} |1\rangle_j)$$

with a probability $p_\phi$. This yields the ensemble $\{|\psi\rangle_\phi, p_\phi\}$ which may be equivalently expressed by the mixed state

$$\rho = \sum_\phi p_\phi |\psi\rangle_\phi \langle \psi|$$

or in the continuum limit as

$$\rho = \int p_\phi |\psi\rangle_\phi \langle \psi| d\phi.$$
Before integrating note that in matrix form $|\psi\rangle_\phi \langle \psi|$ has the following representation,

$$|\psi\rangle_\phi \langle \psi| \rightarrow \bigotimes_{j=1}^{N} \begin{bmatrix} |a_j|^2 & a_j b_j^* e^{-i\phi} \\ a_j^* b_j e^{i\phi} & |b|^2 \end{bmatrix}. \quad (40)$$

For a Gaussian distribution $p_\phi = (4\pi\alpha^{-\frac{1}{2}}) e^{-\frac{\alpha^2}{2}}$ the above integrals yields

$$\left[ \begin{array}{cc} |a_j|^2 & a_j b_j^* e^{-\alpha} \\ a_j^* b_j e^{-\alpha} & |b|^2 \end{array} \right]. \quad (41)$$

As one can see from the decaying off-diagonal entries, the random dephasing due to the environment leads to decoherence. Since we now know that the environment is decohering our system and we know exactly how the environment is doing it, we have all of the information necessary to find a DFS. For starters let's consider the case $N=2$. The dephasing for each of the constituents of the corresponding Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is summarized by the following.

$$|00\rangle \rightarrow |00\rangle \quad (42)$$

$$|01\rangle \rightarrow e^{i\phi}|01\rangle \quad (43)$$

$$|10\rangle \rightarrow e^{i\phi}|10\rangle \quad (44)$$

$$|11\rangle \rightarrow e^{2i\phi}|11\rangle. \quad (45)$$

The key thing to observe is that the states $|01\rangle$ and $|10\rangle$ interact with the environment in the exact same way, this means that they are degenerate states of the Hamiltonian associated with the dephasing, not presented explicitly here. Let us take $|\psi\rangle$ to be in $Span\{|01\rangle, |10\rangle\}$ and allow it to interact with the same dephasing environment.

$$|\psi\rangle = a|01\rangle + b|10\rangle \rightarrow ae^{i\phi}|01\rangle + be^{i\phi}|10\rangle = e^{i\phi}|\psi\rangle$$

but overall phases are unimportant since they do not change the state of the system. This means that $Span\{|01\rangle, |10\rangle\}$ is a 2D DFS. There are two more DFS, but these are just the spaces $Span\{|00\rangle\}$ and $Span\{|11\rangle\}$ which are trivial. It is easy to generalize this process to higher dimensions. For $N=3$ the largest DFS are $Span\{|001\rangle, |010\rangle, |100\rangle\}$ and $Span\{|011\rangle, |101\rangle, |110\rangle\}$ while the remaining DFS are the trivial spaces $Span\{|000\rangle\}$ and $Span\{|111\rangle\}$. For the general case each basis element of the Hilbert space $\bigotimes_{j=1}^{N} \mathbb{C}^2$, i.e. the space of the system, can be characterized by the number of ones it has. For example, if $N=100$ and we want a basis element with 25 zeros then there are $\binom{100}{25}$ basis elements that we may choose from with this property. Basis elements with the same amount of ones will experience the same overall phase shift from the environmental dephasing and their span will therefore form a DFS. There hence exist $N+1$ DFS for an $N$ qubit system with dimension $\begin{pmatrix} N \\ k \end{pmatrix}$ for $k$ ones. The largest possible DFS therefore has dimension $\binom{N}{N(F_{\frac{\sqrt{N}}{2}})}$, $F(x)$ is the floor function. Since storing the largest amount of information is desired one would of course be interested in the largest DFS possible. An important question to ask is whether or not such a DFS is large enough? Such a question depends on the needs in practice. To give an answer to this question nonetheless note that an upper bound for the dimension of any DFS is the dimension of the qubit system ($2^N$ for $N$ qubits). The biggest DFS will have dimension $\binom{N}{N(F_{\frac{\sqrt{N}}{2}})}$, stirling’s formula now gives $log_2\left(\binom{N}{N(F_{\frac{\sqrt{N}}{2}})}\right) \approx N - \frac{1}{2}log_2(\pi \frac{N}{2})$. Consequently, $max[Dim(DFS)] = \binom{N}{N(F_{\frac{\sqrt{N}}{2}})} \approx 2^{N-\frac{1}{2}log_2(\pi \frac{N}{2})}$ whose relative difference to the dimension of the qubit system is

$$\frac{|max[Dim(DFS)] - 2^N|}{2^N} = \frac{|1 - \sqrt{\pi \frac{N}{2}}|}{\sqrt{\pi \frac{N}{2}}} \stackrel{N\gg 1}{\longrightarrow} 1.$$ 

Although the upper bound $2^N$ is not attainable we can nevertheless obtain a value for $Dim[DFS]$ relatively close to $2^N$ in the large $N$ limit. We conclude that in dephasing models of this sort the DFS are large enough to meet any practical needs as long as $N$ is large.

### 2.4.2 Collective Dephasing

We now consider a concrete("real world") model and its corresponding DFS. Take the system to be a collection of spins and the environment to be a bosonic bath. An interaction Hamiltonian for this type of open system may have the following form.

$$H_I = \sum_{i=1}^{N} \sigma_i^z \otimes \sum_{j} (g_{ij} a_j^\dagger + g_{ij}^* a_j) \quad (46)$$

Here the operators $\sigma_i^z$ and $a_j$ belong to the system and bath algebra respectively. The so called ladder operators $a_j^\dagger$ and $a_j$ are responsible for energy flowing to in and out of the environment, i.e. from and to the spin system. We will again assume
quantum measurement limit in what follows for simplicity. i.e. $H_{\text{tot}} \approx H_I$. Recall that the existence of a DFS is related to the presence of a dynamical symmetry within the Hamiltonian of our open system. In this case $H_I$, approximately. An effective and active way to yield such a symmetry is to require that all of the spins interact with each mode of the environment in the exact same way. $g_{ij}$ in $H_I$ will therefore loose their $i$ dependence affording the Hamiltonian to become separable. $H_I = S_z \otimes E$, $S_z = \sum_i \sigma_z^i$ and $E = \sum_j (g_j a_j^\dagger + g_j^* a_j)$. This limiting case is referred to as collective decoherence. It is a useful idea to strive for in since the DFS for such a setup are identical to the ones in the previous subsection. DFS in this case will be made up of degenerate eigen basis of $S_z$. Let $N = 2$, in this case the following describes the action of $S_z$ on the basis $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$. Here we use 0 to denote spin up and 1 to denote spin down.

$$|00\rangle \rightarrow 2|00\rangle \quad (47)$$

$$|01\rangle \rightarrow |01\rangle - |01\rangle = 0 \quad (48)$$

$$|10\rangle \rightarrow |10\rangle e^{-i\frac{\sigma_z}{\hbar}} |01\rangle = 0 \quad (49)$$

$$|11\rangle \rightarrow -2|11\rangle \quad (50)$$

The largest DFS is that formed by $\text{Span}\{|01\rangle, |10\rangle\}$. Since the DFS structure is identical in this case to that developed in the previous subsection we can also conclude that the dimension of the largest possible DFS get relatively close to the dimension of the spin system in the large N limit. Much of the noise imposed on quantum computers comes from the bosonic baths that the information (system) is interacting with, this could present itself in the form of phonons, photons, both and other ways. Engineering an environment that yields an $H_I$ of the form presented here is therefore of high interest, in the quantum measurement limit, since it provides us with relatively large DFS to store and process our information with. In reality the intrinsic dynamics of the system Hamiltonian will influence the dynamics and the collective decoherence symmetry will at best be accurate up to a small symmetry breaking perturbation. If the perturbation is small and the quantum measuring limit is a good approximation then the approximate DFS obtained in this case can be very robust to decoherence on a long enough time scale.

### 2.4.3 independent decoherence

In the previous subsection we have seen the best case scenario for DFS in an open system consisting of a cohort of qubits interacting symmetrically with a bath. Now we consider the other side of the coin and assume that the coupling terms in (46) $g_{ij}$ to be different for each qubit, this is called independent decoherence. The DFS condition will then require us to find a degenerate N-qubit eigenstate, eigen basis for each qubit operator $\sigma_z^i$. But the only possible DFS are $\text{Span}\{|00\ldots0\rangle\}$ and $\text{Span}\{|11\ldots1\rangle\}$ which are trivial. There are therefore no DFS in the independent decoherence case. Unfortunately, it would seem that most real world applications call for an independent decoherence type of dynamics and we can do little more than find an approximate Hamiltonian yielding symmetries that we may use to generate DFS.

### 3 Master Equations

So far we have tackled open systems by first computing the total system plus environment dynamics, unitary time evolution $U(t)$, and tracing out the environmental degrees of freedom after evolving some initial compound state $\rho_{SE}(0)$. Although the evolution due to $U$ is unitary, tracing over the environmental degrees of freedom will in general yield non-unitary dynamics. Alternatively, we could start with the Liouville – VonNeumann equation (VNE)

$$\frac{\partial}{\partial t} \rho_{SE}(t) = -i\frac{\hbar}{\hbar} [H_{SE}, \rho_{SE}(t)] \quad (51)$$

and solve by first taking the partial trace over the environment which yields a differential equation for the evolution of the reduced density matrix.

$$\frac{\partial}{\partial t} \rho_S(t) = -i\frac{\hbar}{\hbar} Tr_E\{[H_{SE}, \rho_{SE}(t)]\} \quad (52)$$

The partial trace term leads to a non-unitary dynamics which takes the form of a formidable deferential equation which solutions are in general intractable, both analytically and numerically, for any practical case. Fortunately one is usually not interested in the dynamics of the environment, this allows us to focus only on the system and the interaction dynamics. The mathematical treatment of the environment in this case may be of some bath whose dynamics are approximately unperturbed by the system, this type of approximation is referred as the Born approximation. This defines a specific type of open quantum system whose initial and evolved states take the form $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)$ and $\rho_{SE}(t) = \rho_S(t) \otimes \rho_E(t)$ respectively. Now, using this assumption the reduced (VNE) is approximately

$$\frac{\partial}{\partial t} \rho_S(t) = -i\frac{\hbar}{\hbar} Tr_E\{[H_{SE}, \rho_S(t) \otimes \rho_E(0)]\}, \quad (53)$$

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formally integrating this equation we arrive at
\[
\rho_S(t) = \rho_S(0) - \frac{i}{\hbar} \int_0^t dt_1 T_{RE} [H_{SE}, \rho_S(t_1) \otimes \rho_E(0)].
\] (54)

Let us now substitute equation (54) into equation (53).
\[
\frac{\partial}{\partial t} \rho_S(t) = -\frac{i}{\hbar} T_{RE} \{[H_{SE}, \rho_S(0) \otimes \rho_E(0)]\} + \frac{\epsilon^2}{\hbar^2} \int_0^t dt_1 T_{RE} \{[H_{SE}, [H_{SE}, \rho_S(t_1) \otimes \rho_E(0)]]\}
\] (55)

This equation will be referred to as the pre-Markovian equation. We could formally integrate the above and substitute the result back into (53) to obtain a term of order 3 in \( \frac{\epsilon}{\hbar} \), repeating this process indefinitely yields a convoluted iterative solution to the dynamics. We will not bother expanding the series completely and will stop at (55) and use such an equation to derive the so called Born – Markov Master equation (BMME) which is a local in time version of (55) obtained from appropriate approximations.

### 3.1 Spontaneous emission

We further develop the theory of non-unitary dynamics and decoherence, in particular those governed by a BMME via an example. Consider a two-level atom coupled to a bath and let the Hamiltonian
\[
H_{SE} = \frac{\hbar \omega_a}{2} \sigma_z + \hbar \sum_k \omega_k b_k^\dagger b_k + \hbar \sum_k (g_k b_k + g_k^* b_k^\dagger)(\sigma_+ + \sigma_-) = H_S + H_E + H_I
\] (56)

describe the dynamics of the total system. Here we use the convention \( A \otimes B := AB \) for the sake of visual clarity. \( H_S = \frac{\hbar \omega_a}{2} \sigma_z \) is the self Hamiltonian of the system, \( H_I = \sum_k (g_k b_k + g_k^* b_k^\dagger)(\sigma_+ + \sigma_-) \) is the interaction Hamiltonian and \( H_E = \hbar \sum_k \omega_k b_k^\dagger b_k \) is the bath Hamiltonian of the environment. Before moving on, we will switch to the interaction picture in order to simplify things a bit. This allows us to trade in the equation
\[
\frac{\partial}{\partial t} \rho_{SE}(t) = -\frac{i}{\hbar} [H_{SE}, \rho_{SE}(t)]
\] (57)

for the equivalent
\[
\frac{\partial}{\partial t} \rho_{int}(t) = -\frac{i}{\hbar} [H_{int}, \rho_{int}(t)].
\] (58)

The new figures are defined below.

- \( \rho_{int}(t) = e^{\frac{i}{\hbar}(H_S+H_E)t} \rho_{SE}(t) e^{-\frac{i}{\hbar}(H_S+H_E)t} \)
- \( H_{int} = e^{\frac{i}{\hbar}(H_S+H_E)t} H_I e^{-\frac{i}{\hbar}(H_S+H_E)t} \)

The operator \( H_{int} \) simplifies to \( \hbar \sum_k g_k(b_k \sigma_+ e^{i(\omega_a+\omega_k)} + b_k^\dagger \sigma_- e^{-i(\omega_a+\omega_k)} + b_k \sigma_+ e^{-i(\omega_a+\omega_k)} + b_k^\dagger \sigma_- e^{i(\omega_a+\omega_k)}) \) with some calculation. In systems of this sort it is often reasonable to remove the terms involving \( \omega_a + \omega_k \) in the argument and retain only the terms involving \( \omega_a - \omega_k \). \( \omega_a + \omega_k \) is expected to be much larger than \( \omega_a - \omega_k \), more can be said about this but it would go beyond the scope of this paper so we refer the interested reader to the excellent paper on the subject by Fuji [5] or Townsend’s text on Quantum Mechanics for an exposition[6]. This type of approximation is known as the Rotating Wave Approximation and it merits use on the grounds that the terms involving \( \omega_a + \omega_k \) in the argument simply average to zero in the time frame we are interested in and can thus be ignored without a significant contribution to error.

\[
H_{int} \approx \hbar \sum_k g_k(b_k \sigma_+ e^{i(\omega_a-\omega_k)} + b_k^\dagger \sigma_- e^{-i(\omega_a-\omega_k)}).
\] (59)

We may now take our approximated \( H_{int} \) and plug it into equation (55) assuming that the Born approximation is also applicable, indeed it is since a sea of oscillators can hardly be influenced by a single two-level atom. Our pre-Markovian equation in this case is
\[
\frac{\partial}{\partial t} \rho_S(t) = -\frac{i}{\hbar} T_{RE} \{[\hbar \sum_k (a_k(t)b_k \sigma_+ + a_k^*(t)b_k^\dagger \sigma_-), \rho_S(0) \otimes \rho_E(0)]\}
\]
\[
\quad + \frac{\epsilon^2}{\hbar^2} \int_0^t dt_1 T_{RE} \{[\hbar \sum_k (a_k(t)b_k \sigma_+ + a_k^*(t)b_k^\dagger \sigma_-), [\hbar \sum_k (a_k^*(t_1)b_k \sigma_+ + a_k^*(t_1)b_k^\dagger \sigma_-), \rho_S(t_1) \otimes \rho_E(0)]\}.
\] (60)

\( \rho_S(t) \) is actually \( \rho_{S_{int}}(t) \) but we remove the second subscript for clarity and \( a_k \) is defined as \( g_k e^{i(\omega_a-\omega_k)} \). To move forward an initial state must be specified, let \( \rho_{SE}(0) = \rho_S(0) \otimes |\Omega\rangle_E \langle \Omega| \). The system’s initial state is yet to be defined and the
environment is in the vacuum state, i.e. ground state. We can compute $\text{Tr}_E\{[\sum_k (a_k(t)b_kσ_+ + a_k^†(t)b_k^†σ_-), ρ_S(0) \otimes |\Omega⟩⟨\Omega|]\}$ with relative ease. Using linearity of the commutator and some properties of the tensor product, note that we have been using the short hand $b_kσ_+$ when referring to $b_k \otimes σ_+$, we can rewrite the above commutator as follows.

$$\sum_k [σ_+ + σ_-, ρ_S(0)]\text{Tr}_E\{[a_k b_k + G_k^*b_k^†, |\Omega⟩⟨\Omega|]\}.$$  \hspace{1cm} (61)

But $\text{Tr}_E\{[a_k b_k + a_k^†b_k^†, |\Omega⟩⟨\Omega|]\} = \sum_i ⟨E_i | [a_k b_k + a_k^†b_k^†, |\Omega⟩⟨\Omega|]|E_i⟩$. The only term in the trace that is non zero is the summand originating from the vacuum element. The trace therefore simplifies to

$$⟨\Omega|a_k b_k|Ω⟩ − ⟨Ω|a_k b_k|Ω⟩ + ⟨Ω|a_k^†b_k^†|Ω⟩ − ⟨Ω|a_k^†b_k^†|Ω⟩,$$

which is zero since $b|Ω⟩ = 0$ and $⟨Ω|b^†⟩ = 0$.

Furthermore it can be shown that the integral term in equation (60) can be reduced to

$$\int_0^t dt_1 \sum_k \text{Tr}_E\{[[a_k(t)b_kσ_+ + a_k^†(t)b_k^†σ_-), [a_k(t_1)b_kσ_+ + a_k^†(t_1)b_k^†σ_-], ρ_S(t_1) \otimes ρ_E(0)]\}.$$ \hspace{1cm} (62)

since only $k = k'$ terms are non zero. Now we come face to face with a technical hurdle, we must now compute the 16 partial traces originating from the integrand in the above. We will simply proceed to the result but will display the fully expanded integrand below just to invoke appreciation for the level of complexity that arises in quantum open systems, even when the system, environment and the dynamics are as simple as we laid them out to be. The spontaneous emission model is virtually as simple as a quantum open system can get without becoming trivial.

$$\text{Tr}_E\{[[G_k^*t)b_kσ_+ + G_k^†t)b_k^†σ_-], [G_k^*t_1)b_kσ_+ + G_k^†t_1)b_k^†σ_-], ρ_S(t_1) \otimes ρ_E(0)]\} =$$

$$= \text{Tr}_E(G_k^*t)G_k(t_1)b_k^2σ_+^2ρ_S(t_1) \otimes ρ_E(0)) − \text{Tr}_E(G_k^*t)G_k(t_1)b_kσ_+ρ_S(t_1) \otimes ρ_E(0)b_kσ_+ +$$

$$− \text{Tr}_E(G_k^*t)G_k(t_1)b_kσ_+ρ_S(t_1) \otimes ρ_E(0)b_kσ_+ + \text{Tr}_E(G_k^*t)G_k(t_1)ρ_S(t_1) \otimes ρ_E(0)b_k^2σ_+^2 +$$

$$+ \text{Tr}_E(G_k^*t)G_k(t_1)b_k^2σ_+ρ_S(t_1) \otimes ρ_E(0)) − \text{Tr}_E(G_k^*t)G_k(t_1)b_k^2σ_+ρ_S(t_1) \otimes ρ_E(0)b_kσ_+ +$$

$$− \text{Tr}_E(G_k^*t)G_k(t_1)b_k^2σ_+ρ_S(t_1) \otimes ρ_E(0)b_kσ_+ + \text{Tr}_E(G_k^*t)G_k(t_1)b_kσ_+ρ_S(t_1) \otimes ρ_E(0)b_k^2σ_+ +$$

$$+ \text{Tr}_E(G_k^*t)G_k(t_1)b_kσ_+ρ_S(t_1) \otimes ρ_E(0)b_kσ_+ + \text{Tr}_E(G_k^*t)G_k(t_1)b_k^2σ_+ρ_S(t_1) \otimes ρ_E(0)) −$$

$$− \text{Tr}_E(G_k^*t)G_k(t_1)b_kσ_+ρ_S(t_1) \otimes ρ_E(0)b_kσ_+ + \text{Tr}_E(G_k^*t)G_k(t_1)b_k^2σ_+ρ_S(t_1) \otimes ρ_E(0)b_kσ_+ +$$

Collecting the nonzero terms in the final expression one gets

$$\frac{∂}{∂t}ρ_S(t) = − \int_0^t dt_1 \sum_k a_k(t)a_k^†(t_1)(σ_+σ_−ρ_S(t_1) − σ_−ρ_S(t_1)σ_+) + h.c.$$ \hspace{1cm} (63)

- h.c. is the Hermitian conjugate of the neighboring summand.

The term $\sum_k a_k(t)a_k^†(t_1)$ and its conjugate can be approximated by an integral due to our assumptions that the bath is large relative to our system and the system interacts equally with each of the virtually continuous degrees of freedom of the environment. Recalling how $a_k(t)$ was defined, our sum is approximated by

$$Γ(t − t_1) = \int_0^∞ dω P(ω)g^2(ω)e^{i(ω_ω − ω)(t − t_1)}.$$  \hspace{1cm} (64)

Here $P(ω)$ is the density of the field modes as a function of frequency, and in practice the term $P(ω)g^2(ω)$ is smoothly varying in the vicinity of $ω_ω$, and therefore our integrand $Γ(t − t_1)$ is sharply peaked at $t = t_1$, this is an adaption of the adaptation Wise et al man give in their book [7] on quantum control. This means that we can approximate the term $ρ_S(t_1)$ by $ρ_S(t)$. The importance of this approximation just made here can not be overstated, this is what we have been working towards, it means that the integral term in our equation is now local in time. This is known as the Markov approximation.

$$\frac{∂}{∂t}ρ_S(t) = (σ_+σ_−ρ_S(t) − σ_−ρ_S(t)σ_+) \int_0^t dt_1 Γ(t − t_1) + h.c.$$ \hspace{1cm} (64)

We changed the lower limit of integration to $−∞$ since the integrand $Γ(t − t_1)$ is negligible for values of $t_1$ varying significantly from $t$. I will make the substitution $Δω_a = i^2 = i \int_0^∞ dt_1 Γ(t − t_1)$ where $Δω_a$ is the frequency shift of the two level atom and $γ$
is the radioactive decay of the atom [6]. With the above substitution and the definition $D[\sigma_{-}]\rho = \sigma_{-}\rho\sigma_{+} - \frac{1}{2}(\sigma_{+}\rho\sigma_{+} + \sigma_{-}\rho\sigma_{-})$ we can do one last modification to our differential equation, i.e.

$$
\frac{\partial}{\partial t}\rho_S(t) = -\frac{i}{2}\Delta\omega_x[\sigma_z, \rho_S(t)] + \gamma D[\sigma_{-}]\rho_S(t).
$$

(65)

We may now revert back from the interaction picture.

$$
\frac{\partial}{\partial t}\rho_S(t) = -\frac{i}{2}(\omega_a + \Delta\omega_a)[\sigma_z, \rho_S(t)] + \gamma D[\sigma_{-}]\rho_S(t).
$$

(66)

An equation of this form is called a Born – Markov Master equation, Lindblad Master equation and/or an equation of Lindblad form.

### 3.1.1 Solution

The most general form a the state matrix $\rho_S(t)$ can take is

$$
\rho_S(t) = \frac{1}{2}[I_2 + x(t)\sigma_x + y(t)\sigma_y + z(t)\sigma_z]([7]\text{chapter 3}).
$$

(67)

Coherences are present via the $\sigma_y$ and $\sigma_z$ terms, we will therefore monitor decoherence via $x(t)$ and $y(t)$. The scalar functions $x(t)$, $y(t)$, and $z(t)$ are computed in the following way.

- $\frac{\partial}{\partial t}z(t) = Tr(\sigma_x \frac{\partial}{\partial t}\rho_S(t))$
- $\frac{\partial}{\partial t}y(t) = Tr(\sigma_y \frac{\partial}{\partial t}\rho_S(t))$
- $\frac{\partial}{\partial t}x(t) = Tr(\sigma_z \frac{\partial}{\partial t}\rho_S(t))$

Using the Lindblad Master equation to substitute for $\frac{\partial}{\partial t}\rho_S(t)$ these equations become

- $\frac{\partial}{\partial t}z(t) = -\gamma(z(t) + 1)$
- $\frac{\partial}{\partial t}y(t) = (\Delta\omega_a)x(t) - \frac{\gamma}{2}y(t)$
- $\frac{\partial}{\partial t}x(t) = -(\Delta\omega_a)y(t) - \frac{\gamma}{2}x(t)$

with solutions

- $z(t) = 2e^{-\gamma t} - 1$
- $y(t) = -e^{-\frac{\gamma}{2}t}\sin((\omega_a + \Delta\omega_a)t)$
- $x(t) = e^{-\frac{\gamma}{2}t}\sin((\omega_a + \Delta\omega_a)t)$.

The solution to the Lindblad Master equation in the spontaneous emission case is therefore the following state matrix.

$$
\rho_S(t) \rightarrow \begin{bmatrix}
\frac{e^{-\gamma t}}{1 - e^{-\gamma t}} & e^{-\frac{\gamma}{2}t}\sin((\omega_a + \Delta\omega_a)t)(1+i) \\
1 - e^{-\gamma t} & 1 - e^{-\gamma t}
\end{bmatrix}
$$

(68)

As is evident from Figure 2, there is both dissipation and decoherence present. For $t$ significantly larger that $\frac{1}{\gamma}$ almost certainly absorbs the energy of the atom, destroying the coherences in the process.

### 3.2 Dynamical maps and semigroups

In section 3.1 we executed a method very different for solving the Schrödinger than what was elaborated upon in section 2.2.2 where we attained the following formal solution for an arbitrary open quantum with the initial total state $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)$.

$$
\rho_S(t) = \sum_{ij} p_{ij}\langle E_j|U(t)|E_i\rangle\rho_{S}(0)\langle E_i|U(t)|E_j\rangle.
$$

(69)

A mapping defined as above,

$$
\nu_t\rho_S(0) := \sum_{ij} p_{ij}\langle E_j|U(t)|E_i\rangle\rho_S(0)\langle E_i|U(t)|E_j\rangle.
$$

(70)

is an instance of a dynamical map.
Definition 5. Dynamical map (informal definition): $\nu_t : \rho(0) \to \rho(t)$ is said to be a dynamical map if it takes an arbitrary quantum state $\rho(0)$ to a final quantum state $\rho(t)$ at some fixed $t$ in accordance with the rules of quantum mechanics.

Recall that (70) is obtained simply via the unitary evolution of the compound system and a partial trace over environmental degrees of freedom. Schrödinger’s equation is responsible for the dynamics and therefore (70) is in accord with the rules of quantum mechanics. The Krauss operators $\{E_j U(t) | E_j\}$ from now on written as $K_{ji}(t)$ are trace preserving operators over the Banach space of statistical operators, i.e. trace class operators with trace one, satisfying the completeness constraint

$$\sum_{ij} W_{ij}(t) W_{ji}^\dagger(t) = I_S.$$  

The number of Kraus operators required to represent a dynamical map will always be bounded by $\text{dim}(\mathcal{H}_S)^2$. The most general way an open quantum system may evolve is expressed by (70), therefore any dynamical map can be completely characterized in terms of a set of Kraus operators $\{K_{ij}\}_{ij}$. The description given for dynamical maps is very informal and will be the one used in the remainder of this paper, a more formal definition is provided but not elaborated upon. In what follows $\mathcal{D}(\mathcal{H}_S)$ is the space of statistical operators, i.e. density matrices of the system $S$.

Definition 6. Dynamical map (formal) A map $\nu_t : \mathcal{D}(\mathcal{H}_S) \to \mathcal{D}(\mathcal{H}_S)$ is a dynamical map iff it is a completely positive map, has convex linearity and is trace preserving.

The family of dynamical maps $\{\nu_t | t \geq 0\}$ satisfying the semigroup condition $\nu_s \nu_t = \nu_{s+t}$ forms what is referred to as a quantum dynamical semigroup. It can be shown that, under certain assumptions, there exists an operator $\mathcal{L}$, as illustrated in Alicki et al 2007 [8], such that

$$\nu_t = e^{\mathcal{L}t}. \quad (72)$$

We have cut many corners, a rigorous development of dynamical maps and semigroup theory would require another paper, but the informal notions presented can be reconciled with the work in (3.1) on spontaneous emission by noting that

$$\frac{\partial}{\partial t} \rho_S(t) = \frac{\partial}{\partial t} \nu_t \rho_S(0) = \frac{\partial}{\partial t} e^{\mathcal{L}t} \rho_S(0) = \mathcal{L} \nu_t \rho_S(0) = \mathcal{L} \rho_S(t). \quad (73)$$

$\mathcal{L}$ is indeed just the Lindbladian, we can therefore conclude that the Lindbladian is the generator of the dynamical semigroup $\{\nu_t | t \geq 0\}$. This yields the master equation! The spontaneous emission case is therefore completely characterized by dynamical maps and the dynamics are generated by the dynamical semigroup $\{e^{\mathcal{L}t} : t \geq 0\}$ where $\mathcal{L} = \frac{1}{\hbar} \sum_{i} \sum_{\delta} \omega_{i}\sigma_{\delta} + \gamma D[\sigma_{-}]$. For a general finite dimensional Hilbert space $\mathcal{H}_S = N$, $\text{dim} \mathcal{H}_S$, the most general form of $\mathcal{L}$ is

$$\mathcal{L} \rho_S(t) = -\frac{i}{\hbar} [H, \rho_S(t)] + \sum_{i,j} \alpha_{ij}(t) \{F_i \rho_S(t) F_j^\dagger - \frac{1}{2} F_j^\dagger F_i \rho_S(t) - \frac{1}{2} \rho_S(t) F_j^\dagger F_i\}. \quad (74)$$

The operators $F_i$ are a set of $N^2$ linear operators forming an orthonormal basis for the space $\mathcal{B}(\mathcal{H}_S)[9]$, one of these elements will be proportional to the identity. More examples of Markovian master equations will be presented in the following chapter.
3.3 Non-Markovian Dynamics

The assumptions that lead to the validity of the Bork-Markov master equation are not in general applicable and there are indeed a slew of interesting physical systems with very relevant Markovian dynamics. The most general case

\[ \frac{\partial}{\partial t} \rho_S(t) = \mathcal{J}[\rho_S(t), t], \]

where \( \mathcal{J} \) is a superoperator that takes the history of the reduced density matrix as input, is extremely difficult to solve at best and intractable in the worst case. When the Markov approximation does not apply, there exists another factor to consider. One may approximate (75) with

\[ \frac{\partial}{\partial t} \rho_S(t) = \mathcal{J}^n(t) \rho_S(t) \]

where the operator \( \mathcal{J} \) is now only locally dependent on time. This is usually obtained via modifications to the appropriate Born-Markov equation.

4 Canonical Decoherence Models

We have already seen an example of a canonical decoherence model, i.e., the spontaneous emission model which can be applied in a more general manner to a variety of two level ("Spin") systems interacting with a bosonic environment under the quantum measurement limit constraints outlined in chapter 3. Canonical decoherence models of this type are referred to as Spin – Boson decoherence models. There are other so-called "canonical decoherence" models which are of great interest in practice. These are Quantum Brownian motion, Spin – Spin and collisional decoherence models. In this section we will introduce the respective master equations and study the decoherence, dissipation (if any), and time scales corresponding to each system.

4.1 Collisional Decoherence

Let \( |x⟩ \) represent the position eigen state of a mass point representing some macroscopic/mesoscopic object/system. \( |E⟩ \) will represent the state of a scattered particle or particles/environment. We will consider a simple case where the mesoscopic particle that plays the role of our system \( S \) experiences no recoil due to the scattered particles. Examples of systems with no recoil would be any large molecule, dust particle, or a bowling ball scattering photons. The photons in all of these cases virtually have no effect on the trajectory of the systems and a recoil less approximation is therefore appropriate. The scattering dynamics are summarized by the appropriate \( S – matrix \) (scattering matrix) of the recoilless limit like so.

\[ |x⟩|E⟩ \rightarrow |x⟩|E⟩ = |x⟩S_x|E⟩. \]

The \( S – matrix \) \( S_x \) is a unitary operator that maps some initial state of incoming particles, i.e., photons, to the final state, scattered state. This \( S – matrix \) indeed depends on the type of particles being scattered off the center \( x \), the type of particle doing the scattering, the forces involved and the initial velocities. We will not go into those details here. Joos and Zeh in their 1985 seminal paper [10] considered such a scattering set up in order to compute decoherence time scales for mesoscopic systems in a super position state. In their paper an arbitrary wave function of their total system evolves as depicted below.

\[ \int d^3 x \phi(x)|x⟩|E⟩ \rightarrow \int d^3 x φ(x) φ(x')^* \rho_S(x, x')|E⟩S_x^† S_x|E⟩. \]

the reduced density matrix is therefore

\[ ρ_S(x, x') = φ(x) φ(x')^* \rightarrow ρ_S(x, x')|E⟩S_x^† S_x|E⟩. \]

Here \( φ(x) \in \mathcal{H}_S \) and \( |E⟩ \in \mathcal{H}_E \), and \( ρ_S(x, x') \) are the matrix elements of \( ρ_S \).

4.1.1 Long-Wavelength limit

Let us fix \( x \) and \( x' \) for the moment. If we assume that the wavelength of the incoming photons satisfies \( λ >> |x – x'| \) then a single scattering event will not resolve the distance \( |x – x'| \), i.e., coherences of distances on this order will not instantaneously disappear. They will decay exponentially. Using photons restricted to the above condition it can be shown that

\[ \langle E | S_x^† S_x | E⟩ \approx e^{-L(x-x')^2} \]

for \( n \) photon scattering events. Time appears due to its relation to number of scattering events \( n \). The relationship is the following, \( t = \frac{n}{L^2} \), \( L \) is the length used to normalize the momentum wave functions [10]. Just as a sanity check note that \( t \) increases as \( n \) increases. Joos and Zeh basically count time in terms of scattering events. This is reasonable if the light is...
continuously flowing toward our particle. The constant $\Lambda$ is the scattering constant that represents the physical properties of the system-environment interaction. This constant is larger for larger systems, i.e. a bowling ball will have a $\Lambda$ that is significantly greater than the $\Lambda$ of a dust particle. The computation of $\Lambda$ is omitted due to the tedium it provides, this computation too is carried out for the long-wavelength limit in Joos and Zeh’s paper \[10\]. Equation (79) may now be reduced to

$$\rho_s(x, x') \xrightarrow{t} \rho_s(x, x') e^{-\Lambda t(x-x')^2}. \quad (81)$$

This implies the following equation

$$\rho_s(x, x', t) = e^{-\Lambda t(x-x')^2} \rho_s(x, x', 0). \quad (82)$$

A time derivative of the latter yields the very important deferential equation

$$\frac{\partial}{\partial t} \rho_s(x, x', t) = -\Lambda(x - x')^2 \rho_s(x, x', t). \quad (83)$$

This equation may now be written in the full density matrix form as

$$i\hbar \frac{\partial}{\partial t} \rho_S(t) = -i\Lambda [x, [x, \rho_S]]$$

. This is indeed the nonunitary part of the master equation of the recoilless scattering model. As evidence look at the solution (82), clearly off-diagonal entries are decaying. Since the scattering process does not affect the trajectory of our system particle we may include the intrinsic dynamics $H_S = \frac{p^2}{2m}$ into the latter equation to render the full master equation

$$i\hbar \frac{\partial}{\partial t} \rho_S(t) = [\frac{p^2}{2m}, \rho_S(t)] - i\Lambda [x, [x, \rho_S(t)]] \quad (84)$$

Values for $\Lambda$ are given in figure 1 for two differently sized dust particles undergoing scattering interactions with varying environments.

Table 1: $\Lambda$ for two sizes of ”dust particles” and various types of scattering processes according to (82)( from Joos and Zeh 1985). This quantity measures how fast interference between different positions disappears for the long-wavelength limit.

<table>
<thead>
<tr>
<th>Environment</th>
<th>Dust grain, $10^{-3} cm$</th>
<th>Dust particle, $10^{-5} cm$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cosmic background radiation</td>
<td>$10^6$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>300k photons</td>
<td>$10^{19}$</td>
<td>$10^{12}$</td>
</tr>
<tr>
<td>sunlight on earth</td>
<td>$10^{21}$</td>
<td>$10^{17}$</td>
</tr>
<tr>
<td>Air molecules</td>
<td>$10^{36}$</td>
<td>$10^{32}$</td>
</tr>
<tr>
<td>Laboratory vacuum</td>
<td>$10^{23}$</td>
<td>$10^{19}$</td>
</tr>
</tbody>
</table>

Prior to scrutinizing equation (84) further let use refocus our attention back to equation (82), this encapsulates the non-unitary dynamics of recoilless processes. As expected, the recoilless aspect of this interaction means that there is no dissipation; but as we have already mentioned, this does not mean that decoherence is not present. It is visually clear from the exponential term that the of diagonal terms $x \neq x'$ experience monotonic damping as $t$ becomes large. To further elucidate this let us take a generic state for some mesoscopic system to be in a simple superposition.

$$\phi(x) = N_1 e^{-(x-a_1)^2} + N_2 e^{-(x-a_2)^2} \quad (85)$$

We use Gaussians rather than delta functions as might have been expected due to the choice for eigen kets $|x\rangle$. Although very little harm is done considering $|x\rangle$ an eigen ket the simple truth is that it is not. The Hilbert space appropriate for our mesoscopic particle is $L^2(\mathbb{R})$ and delta functions are not elements of this space. Gaussians live in $L^2(\mathbb{R})$ and better represent the physical fact that there exists resolution limits. $\phi(x)$ is therefore a realistic representation of what a superposition of two positions $a_1$ and $a_2$ might actually be perceived to be in an experiment. The physical motivation aside this Gaussian superposition is a perfectly legitimate state for our particle in a theoretic sense and can be used, despite its simplicity, to illustrate the decohering effects of recoilless scattering for any general state.

In figure 2 we depict the time evolution of a Gaussian superposition. Note the decaying in the off-diagonal entries $\rho_S(x, x', t)$. The decoherence time scale $\tau_{\Delta x} = \frac{1}{\Lambda |\Delta x|^2} |\Delta x = |x - x'|$ which encapsulates the speed in which decoherence takes place, becomes smaller as $\Lambda$ becomes larger. The larger the particle is the larger $\Lambda$ is and therefore the smaller $\tau_{\Delta x}$ is. Notice the differences in the values for $\Lambda$ of dust particles of diameter $10^{-3} cm$ and $10^{-5} cm$ presented in table 1. The values for $\tau_{\Delta x}$, in seconds, pertaining to the larger dust particle case are at least four orders of magnitude larger than the corresponding values pertaining to the smaller dust particle. In table two we analyze at a single size for dust particle under the same conditions but this time login values of $\tau_{\Delta x}$ for a fixed value of $\Delta x$. Notice how fast these coherences dissipate,
Figure 3: From Schlosshauer’s paper “Decoherence 2019”. Progressive decoherence (left to right) for \( \rho_S(x, x', t) = e^{-\Lambda(x-x')^2}\phi^*(x)\phi(x') \) as generated by the master equation (83). Spatial coherence is represented by the along the off-diagonal direction \( x = x' \), becomes damped by the environmental scattering.

Figure 4: From Schlosshauer’s paper “Decoherence 2019”. Collisional decoherence of a density matrix representing a gaussian wave packet as generated by equation (83).

even a vacuum would decohere any positional coherences of sizes comparable to the size of our particle in \( 10^{-14} \) seconds, and these are decoherence time scales for just dust particle of size \( 10^{-3} \) cm. Decoherence time scales for objects much larger, or much more classical, such as a baseball will be many orders of magnitude smaller. This is consistent with our day to day experience in our classical world, we never perceive any macroscopic object or mesoscopic object to be in a superposition because these superpositions are extremely fragile and dissipate, when present, in a time scale that is so much smaller than what we can keep track off with our naked eye.

Table 2: \( \tau_{\Delta x} \) for a large dust particle , \( \Delta x \) is equal to the diameter of the particle and various types scattering processes are considered. These values are obtained from Joos and Zeh 1985 paper.

<table>
<thead>
<tr>
<th>Environment</th>
<th>Dust grain, ( 10^{-3} ) cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>cosmic background radiation</td>
<td>1</td>
</tr>
<tr>
<td>Photons at room temperature</td>
<td>( 10^{-18} )</td>
</tr>
<tr>
<td>Best laboratory vacuum</td>
<td>( 10^{-14} )</td>
</tr>
<tr>
<td>Air at normal pressure</td>
<td>( 10^{-31} )</td>
</tr>
</tbody>
</table>

Let us now return to equation (84) which more generally describes the dynamics of a free particle undergoing scattering interactions with some bosonic environments. As we already showed, the non-unitary term will quench the off-diagonal terms of the state matrix \( \rho_S(t) \) as time progresses, the unitary evolution term \( \left[ \frac{\hat{p}^2}{2m}, \rho_S(t) \right] \) will be responsible for the free spreading of the wave packet. It can be shown that the free particle Hamiltonian has the effect of spreading localized wave packets in \( x \) ( [6] chapter 6) into highly localized ones. As seen in figure 4, a symmetric Gaussian initial state will become flat in the off-diagonal entries \( x = -x' \) and will extend along the diagonal \( x = x' \) which represents the probability distribution of our particles position \( P(x, t) := \rho_S(x, x, t) \).

4.1.2 Short-wavelength limit

If the wavelength of the scattered environmental particles is much smaller than a coherent separation \( \Delta x = |x - x'| \) then these environmental particles can resolve such a separation in a single scattering event, leading to maximum spatial decoherence per scattering event. In this limit the decay of the coherence terms of the density matrix will not depend on position.

\[
\rho_S(x, x', t) = \rho_S(x, x', 0)e^{-\Gamma_{\text{tot}}t}
\]

(86)

this form should not come to us as a surprise since we have already seen it when dealing with spontaneous emission. It was not elaborated upon in chapter 3 but environmental modes near resonance contributed the most to the interaction dynamics between the two level atom and the bosonic environment open system. This led us to a similar decaying off-diagonal term.
different only in the fact that it was oscillatory, otherwise independent of anything but time. In general a scattering process like what he have developed may ultimately divide itself into two cases, one for positional coherences with distances falling under both the short-wavelength and one for those falling under the long-wavelength limit criteria.

4.2 Quantum Brownian Motion

Possibly the most celebrated decoherence model is Quantum Brownian motion. This is a model describing the dynamics of a particle weakly coupled to a thermal bath of noninteracting harmonic oscillators. The self-Hamiltonian of the environment is

\[ H_E = \sum_i \left( \frac{1}{2m_i} p_i^2 + \frac{1}{2} m_i \omega_i^2 q_i^2 \right), \]  

(87)

where \( m_i \) and \( \omega_i \) are the mass and natural frequency of the \( i \)th oscillator, and \( q_i \) and \( p_i \) denote the canonical positions and momenta operators. The interaction Hamiltonian is taken to be

\[ H_I = x \otimes \sum_i c_i q_i, \]  

(88)

a bilinear coupling of the system’s position \( x \) to the positions \( q_i \) of the environmental oscillators. Finally, we will assume that the particle (the system) will have oscillatory intrinsic dynamics,

\[ H_S = \frac{1}{2M} p^2 + \frac{1}{2} M \Omega^2 x^2, \]  

(89)

where \( M \) is the mass of the particle and \( \Omega^2 \) is its natural frequency. Using the Born and Markov approximations as well as some approximations analogous to what was done in the spontaneous emission case it can be shown that the corresponding Master equation is the following [9].

\[ \frac{\partial}{\partial t} \rho_S(t) = -\frac{i}{\hbar} [H_S + \frac{1}{2} M \Delta^2 x^2, \rho_S(t)] - \frac{i \gamma}{\hbar} \left[ x, \{ p, \rho_S(t) \} \right] - D \left[ x[x, \rho_S(t)] \right] - \frac{f}{\hbar} \left[ x, [p, \rho_S(t)] \right]. \]  

(90)

The symbol \( \{, \} \) here represents the Poisson brackets and they act on operators as follows, \( \{ A, B \} = AB + BA \). Below is a list defining all of the constants present in the Master equation for Quantum Brownian Motion, where \( J(\omega) \) is the spectral density of the environment.

- \( \nu(\tau) = \int_0^\infty d\omega \nu(\omega) \coth(\frac{\hbar \omega}{2k_B T}) \cos(\omega \tau) \), noise-kernel.
- \( \eta(\tau) = \int_0^\infty d\omega J(\omega) \sin(\omega \tau) \), dissipation kernel.
- \( \Delta^2 = -\frac{2}{\pi^2} \int_0^\infty d\tau \eta(\tau) \cos(\Omega \tau) \), the square of the shifted natural frequency of the particle.
- \( \gamma = \frac{2}{\pi^2} \int_0^\infty d\tau \eta(\tau) \sin(\Omega \tau) \), damping rate due to dissipation effects.
- \( D = \frac{1}{\pi} \int_0^\infty d\tau \nu(\tau) \cos(\Omega \tau) \), scattering constant analogous to \( \Lambda \) in the previous section.
- \( f = -\frac{1}{\pi} \int_0^\infty d\tau \nu(\tau) \sin(\Omega \tau) \), also represents decoherence but usually negligible, specially at high temperatures [9].

We will not attempt solving this Master equation since it is clear from the form and description of constants that it generates dissipation and decoherence. We will however mention the following key result. It can be shown that \( (\Delta x)^2(t) = \frac{\hbar \Omega^2}{2M \omega^2 t} \) [9]. This is the reason such a model is referred to as Quantum Brownian motion, the ensemble width \( \Delta x(t) \) asymptotically scales as \( \sqrt{t} \) which is the scaling behavior seen in classical Brownian motion.

4.3 Spin environments

Since we have already explored a Spin – Boson open system in depth back in chapter 3, we will skip that case and discuss a bit about spin environment open systems where the system is also a spin system. Spin system, \( S \), with spin environment, \( E \), models are good for describing decoherence induced by low temperature environments [11]. In general if a two level system or collection of two level systems interact strongly with some environment at low temperature, i.e. the interaction Hamiltonian is the only non-negligible term in the total Hamiltonian, we may approximate this environment with a collection of spins. One of the most basic spin environment models has already been presented in the context of decoherence free subspaces, namely

\[ H = H_{int} = \frac{1}{2} \sigma_z \otimes \sum_{i=1}^N g_i \sigma_i^z := \frac{1}{2} \sigma_z \otimes E. \]  

(91)
Initially there is a sharp decoherence but then proceeds an oscillatory behavior in the coherences. This is a toy model since in general orders of magnitudes greater than \(N = 14\) spin are expected to be at play in a realistic spin-chain. This Hamiltonian represents the environmental monitoring of the observable \(\sigma_z\) and leads to decoherence in the \(\{|0\rangle, |1\rangle\}\) eigen basis of \(\sigma_z\). It can be shown that the decoherence rate increases exponentially with the number \(N\) of environmental spins. For large \(N\) and a broad class of distributions of the coupling coefficients \(g_i\) the interference damping of the quantum coherences follows Gaussian time dependence \(e^{-\Gamma^2 t^2}\). \(\Gamma\) here depends on the initial state of the environment and the couplings \(g_i\). There may be intrinsic tunneling present, as we have seen with the ammonia molecule earlier in this paper. In which case the total Hamiltonian becomes.

\[
H = H_S + H_{int} = \frac{1}{2} h \Delta_0 \sigma_x + \frac{1}{2} \sigma_z \otimes \sum_{i=1}^{N} g_i \sigma_z^i
\]

(92)

4.3.1 Spin chains

More exotic spin environment open systems are spin-chains. Consider the nearest neighbor interaction spin-chain system with the following Hamiltonian

\[
H = -\frac{1}{2} \sum_{i=1}^{N} h_n \sigma_z^i - \frac{1}{2} \sum_{i}^{N-1} \{g_x^i \sigma_x^i \otimes \sigma_x^{i+1} + g_y^i \sigma_y^i \otimes \sigma_y^{i+1} + g_z^i \sigma_z^i \otimes \sigma_z^{i+1}\}
\]

(93)

the \(g^i\) and \(h^i\) are coupling constants and intrinsic dynamics constants respectively. Any one of the constituents of this spin-chain can be considered as the system, making the remaining spins the environment. Computing the time evolution operator we can evolve an initial state and trace out the environmental degrees of freedom in order to monitor the coherences of the spin we have deemed to be the system of interest. Let us consider an instance where \(N = 14\), \(h^i = 2(\text{Joules})\) for all \(i\), \(g_{x,y,z}^{i} = 0.002(\text{Joules})\) for all \(i\) and \(j\) where \(j\) takes values of \(x\), \(y\) and \(z\). Set the initial state of the total system to be \(\frac{|1\rangle - |0\rangle}{\sqrt{2}} \otimes_{n=1}^{N-1} |0\rangle\). We limit ourselves to 14 spin due to computational limitations. Let use single out the spin on the far left of the chain, \(i = 1\), and trace out all other degrees of freedom, monitor the coherences. The dynamics can be seen in figure 5. Notice that the coherences are not decaying monotonously, they are inact not decaying at all. They are oscillating in a very bizarre way. As the number of spin \(N\) becomes large these coherences are expected to decay as we would expect for a two level atom in a bosonic bath. However, spin environments produce an interesting dynamical artifact that is not elsewhere seen. i.e. if you wait long enough the coherences are expected to return to their initial value. This is argued to be simply a byproduct of the models since finite spin systems like the one scrutinized just now are not very realistic. They can not exist spontaneously, work has to be done in order to maintain a spin-chain. These systems are indeed very unstable and thus the dynamics afforded by (93) in this case are only dominant for a short time. After this window of time, a system such as our spin chain will couple to its environment. Figure 6 zooms in on the approximate time frame for which the Hamiltonian (93) governs the dynamics of the spin chain.
Figure 6: Zooming in to the approximate time frame for which (93) governs the dynamics. As N becomes large we expect this to take exponential form.

4.3.2 Central Spin systems

So far we have taken parts of the spin-chain to be the environment which monitor the remaining part of the chain. We did this in order to motivate spin environment models. Spin chains are actually very similar to another spin environment model often encountered and of great interest in experimental applications. This is the central spin model, this differs from the spin chain in very little, instead of pair wise interaction all of the environmental spins interact with the ”central” spin. This can be swiftly obtained from our spin chain Hamiltonian (93) by making the following change.

\[
H = -\frac{1}{2} \sum_{i=1}^{N} h_n \sigma_i^z - \frac{1}{2} \sum_{i}^{N-1} \{g_x^i \sigma_x^i \otimes \sigma_x^{i+1} + g_y^i \sigma_y^i \otimes \sigma_y^{i+1} + g_z^i \sigma_z^i \otimes \sigma_z^{i+1}\}.
\]  

(94)

4.3.3 Spin chain in a bath

If we take the entire spin chain to be our system and some bosonic field to be the environment we can use our techniques developed for spin-boson quantum open systems. Let us return to the spin-chain that we were just analyzing. Let’s say that there are 10 spins, only 10 due to computational limitations, in the chain and that the constituents interact with each other via (93), on top of this lets assume some interaction with a bosonic environment. Figure 7 shows the dynamics, for initial state \(|\psi(0)\rangle = |000000000\rangle\) of the expectation values of each of the \(\sigma_i^z\). We wont go into the details of what the interaction Hamiltonian is, here we are only interested in demonstrating general dynamics. The dynamics show a convergent behavior for \(<\sigma_i^z>\) similar to the spontaneous emission case. This convergent behavior can further be correlated to decoherence of the spin-chain system.

5 experimental studies of decoherence

Throughout this paper we have developed the theory of decoherence with little mention of experimental studies proving the existence of quantum coherences and/or their exponential decay in the open system treatment. We have also only briefly alluded to their importance in quantum computation and as experimental tools. Quantum coherences and their sensitivity to varying environmental factors have indeed been studied experimentally in many systems. Some examples of systems where decoherence has been monitored include optomechanical systems, photon states in cavity, matter-wave interferometry, SQUIDS( Super Conducting Quantum Inteference Device), Ion traps and many others. In some instances the ability to measure decoherence is only a secondary consequences of such systems. Systems like SQUIDS for example are designed to detect very small magnetic fields to a high level of precision and afford many practical uses. Both Ion traps and SQUIDS are actually used for quantum computation where SQUIDS and Ions function as qubits, the ions being the cubits in the case of the Ion trap.
that the min overlap is obtained with $\Xi = \pi$. The overlap depends both on $N$ and the phase difference $\Xi$. For a fixed $N$ it can be shown that the min overlap is obtained with $\Xi = \frac{\pi}{2}$. Owing to the properties of rubidium atoms, the experiment by Brune et al [12] depicted in figure 8 achieved relatively large phase shifts of up to $\Xi = 0.31\pi$, with mean photon number $N \approx 10$. For these values the overlap is less than $3 \times 10^{-5}$ making the states $|\alpha e^{i\Xi}\rangle$ and $|\alpha e^{-i\Xi}\rangle$ very nearly orthogonal. This means $|\alpha e^{-i\Xi}\rangle$ and $|\alpha e^{i\Xi}\rangle$ act as a meter that encodes which-state information about the eigenstates $|g\rangle$ and $|e\rangle$.

To measure the degree to which the components of the state $|\pm\rangle$ represent a mesoscopically distinguishable state we must consider the overlap between $|\alpha e^{i\Xi}\rangle$ and $|\alpha e^{-i\Xi}\rangle$.

$$|\langle \alpha e^{i\Xi} | \alpha e^{-i\Xi} \rangle|^2 = e^{-4|\alpha|^2 \sin^2(\Xi)}.$$  \hfill (96)

The overlap depends both on $N = |\alpha|^2$ the mean photon number and the phase difference $\Xi$. For a fixed $N$ it can be shown that the min overlap is obtained with $\Xi = \pi$. To measure the degree to which the components of the state $|\pm\rangle$ represent a mesoscopically distinguishable state we must consider the overlap between $|\alpha e^{i\Xi}\rangle$ and $|\alpha e^{-i\Xi}\rangle$.

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Figure 8: Schematic of experiment by Brune et al [13]. Rubidium atom starts in the oven $O$ and then passes through cavity $R_1$ that uses a microwave $\frac{\pi}{2}$ pulse to prepare the atom in a superposition of two circular Rydberg energy eigen states $|g\rangle$ and $|e\rangle$. Next the atom enters the cavity $C$ containing a photon field in a coherent state $|\alpha\rangle$. The field suffers a dispersive phase shift $\Xi$ that depends on the energy state of the atom, leading to the entangling of the atom with the photonic field in a coherent state. i.e. The output is $\frac{1}{\sqrt{2}}(|g\rangle|\alpha e^{-i\Xi}\rangle + |e\rangle|\alpha e^{-i\Xi}\rangle)$. Next the atom goes through another $\frac{\pi}{2}$ microwave pulse in the cavity $R_2$ to further transform the atomic state with another $\frac{\pi}{2}$ microwave pulse, and finally the atom’s energy is measured by the ionization chambers $D_e$ and $D_g$. This disentangles the atom from the field, leaving the latter in superposition $|\pm\rangle = \frac{1}{\sqrt{2}}(|\alpha e^{i\Xi}\rangle \pm |\alpha e^{-i\Xi}\rangle)$, two coherent field states with distinguishable phase. Figure from [9] chapter 6.

Definition 7. Two-Atom Correlation Signal

$$\eta(\tau) := P_{ee}(\tau) - P_{eg}(\tau).$$

$P_{ee}$ is the probability that both atoms are found in the excited state. $P_{eg}$ is the probability that the second atom is found in the excited state given that the first atom emerged in the ground state. In the absence of decoherence $\eta(\tau) = \frac{1}{2}$ and under maximal decoherence we have $\eta(\tau) = 0$. Figure 9 presents the results of Brune et al for the the two cases both with $N = 3.3$ of $\Xi = 0.13\pi$ and $\Xi = 0.31$. The main result here is that longer wait times prior to sending the second atom through the cavities lead to a smaller $\nu(\tau)$, approaching zero for $\tau$ large. This experiment provides proof that decoherence is a continuous process. It also gives insight to the speeds in which decoherence takes place and the dependence of decoherence on environmental properties.

5.2 Matter-wave interferometry

Positional decoherence for large molecules can be experimentally studied with Talbot-Lau interferometers [17]. These interferometers function essentially the same way double slits do in double slit type experiments where electrons are shot through tiny apertures producing an interference pattern. If the wave property of matter is fundamental, all matter should exhibit such interference patterns with use of the correct interferometers. For molecules such as $C_{60}$ and $C_{70}$ fullerenes and large molecular clusters we are not able to use ordinary double slit experiments to peer into the wave nature of the corresponding matter however. This is due to the size of such objects. One instead uses the mentioned Talbot-Lau interferometers as seen in figure 10. Brezerg et al conducted such an interferometery experiment with $C_{70}$ [18]. The results are depicted in figure 11. Where the visibility is defined as follows.

Definition 8. Visibility

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}.$$

The Visibility can be shown to be a measure of the coherence in the positional degrees of freedom of the molecule going through the interferometer. $V$ equal to 100 percent pertains to zero decoherence while $V$ equal to 0 percent pertains to complete decoherence. We see in figure 11 a fringe pattern with visibility $V = 38\%$ fo $C_{70}$, the peaks are separated by a micrometer. These fringes are a by product of the super position principle of quantum mechanics, i.e. the existence of quantum coherence. As such these fringes should also be sensitive to environmental monitoring. The quintessential example of quantum phenomena is that of the traditional double slit experiment and in such an experiment it is known that upon measuring the trajectory of electrons coming out of the slit one collapses the wave function and destroys the fringes that would appear otherwise. Matter-wave interferometry should exhibit similar sensitivities. In order to confirm that the latter is indeed the case, Hornberg et al and Hackermüller et al conducted Talbot-Lau experiments with varying atmospheric pressure for $C_{70}$ fullerenes [19][20]. The greater the atmospheric pressure the more likely it is for a $C_{70}$ molecule to collide with a gas
Figure 9: Decoherence of superpositions of the coherent states $|\alpha e^{i\Xi}\rangle$ and $|\alpha e^{-i\Xi}\rangle$ for two different values of the phase shift $\Xi$ is shown: $\Xi = 0.13\pi$ (circles) and $\Xi = 0.31\pi$ (triangles). The phase difference between the coherent-state components is also visualized in the insets, showing the vector representations of the components. The mean photon number is $N = 3.3$. Solid lines are theoretical predictions [14]. Figure from [15].

Figure 10: Schematic illustration of the Talbot–Lau interferometer used for demonstrating interference patterns for C60 and C70 fullerene molecules, and for studying their decoherence [18]. Molecules emitted from a source are velocity selected and pass through the first grating to produce sufficient beam coherence. The second grating is a diffraction grating implementing the Talbot effect. The third grating acts as a scanning mask for the molecular density pattern subsequently recorded by ionizing and detecting the molecules. Figure from [18].

Figure 11: Fringes for C70 molecules, as reported by Brezger et al. [18]. Figure adapted with permission from Ref. [18]. Measured fringe visibilities were $V = 38$ percent.
particle and therefore leading to greater collisional decoherence. Figure 11 is in accord with our expectations, as the pressure goes up collisional decoherence becomes greater and visibility becomes smaller. Another experiment by Hackermüller exhibits the sensitivity of visibility (coherence) to initial temperatures of the $C_{70}$ molecules [17]. The hotter the $C_{70}$ molecules were at the start the smaller the visibility was. This is also expected since hotter $C_{70}$ means higher energy in the initial state of the $C_{70}$ which increases the probability of the molecule to emit a photon into the environment, spontaneous decay, which we have already seen also induces decoherence. In this case the emitted photons will emit information about the position of the molecule.

5.3 Other experimental studies of decoherence

In this section we only explored two types of experiments that confirm decoherence to be an environment dependent process that takes place in a continuous fashion. There are many other experiments that also confirm this and many of the apparatuses in such experiments such as SQUIDS and Ion traps can actually be used for quantum computation [21][22]. Among the experiments and technologies mentioned in the intro of this chapter there are other prospective areas for the observation of decoherence. The following is a partial list.

- Quantum dots.
- Mechanical quantum resonators.
- Bose-Einstein condensates.

6 Conclusion

Perhaps the most dramatic thing an open quantum systems approach to quantum decoherence affords is an interpretation of quantum theory at odds with the Copenhagen interpretation. With experimental results like those provided in chapter 5 it is hard to not take the open quantum systems approach as the greater truth. With knowledge of the time scales in which decoherence takes place and the sensitivity of any particular quantum systems state to its environment, one may develop high precision tools to counteract decoherence or mitigate it. The study of decoherence is not an end in it of itself, it serves us not only to confirm that the superposition quality of matter is indeed real and ubiquitous but also to help us understand better ways of controlling decoherence for quantum computation schemes. As tools like SQUIDS, Ion traps and matter-wave interferometers become more readily available and decoherence theory further evolves, quantum computation and quantum technologies will evolve in parallel. The possibilities of what the future holds are nothing but exciting.
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